

# CLASSICAL AND QUANTUM FREE MOTIONS IN THE TOMOGRAPHIC PROBABILITY REPRESENTATION

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## Abstract

Based on a geometric picture, the example of free particle motion for both classical and quantum domains is considered in the tomographic probability representation. Wave functions and density operators as well as optical and symplectic tomograms are obtained as solutions of kinetic classical and quantum equations for the state tomograms. The difference of tomograms of free particle for classical and quantum states is discussed.

## 1 Introduction

The geometrical methods to study the behaviour of classical systems [1] play an important role in better understanding the classical motion and structure of the Hamiltonian and Lagrangian descriptions. Also the geometrical picture of quantum mechanics [2] and a clarifying geometrical sense of calculations in quantum domain shed new light onto intuitively difficult quantum mechanical concepts of states and observables.

One of the applications of a geometrical view on the notion of quantum states is the quantum tomography (see recent reviews [3, 4, 5]). The quantum tomography theory appeared in the connection with experiments in quantum optics [6] where the relation of optical tomograms [7, 8] to the Wigner function given by integral Radon transform was used to measure a photon quantum state identified with the state Wigner function. Optical tomograms as well as symplectic tomograms are fair probability distribution functions. They were suggested [9] to be considered as a primary notion of the quantum state, since the tomograms contain complete information on quantum states as wave functions and density operators. This property of the tomograms turns out to be useful also in classical domain, where the tomograms can be employed as an alternative to the probability density on the phase space describing the states of classical particle in classical statistical mechanics [10].

Bearing in mind the geometrical considerations of quantum and classical motions developed in [1, 2], the aim of this paper is to give a short review of results obtained in the tomographic probability description of classical and quantum states using the example of the motion of a one-dimensional free particle of mass one. In Sec. 2, we review free particle motion in classical and quantum domains in the standard classical and quantum mechanics. In Sec. 3, we discuss the tomographic-probability representation of classical and quantum free motion and present the basic equations. In Sec. 4, we consider the difference of classical and quantum state tomograms. We conclude in Section 5.

## 2 Free motion

### 2.1 Classical mechanics

Let us recall the description of the free motion of a one-dimensional particle in classical mechanics, when no fluctuations of position  $q$  and momentum  $p$  are present. The Hamiltonian reads (we take the mass  $m = 1$ ) :

$$H = \frac{p^2}{2}. \tag{1}$$

The trajectories determined by this Hamiltonian are straight lines defined by two integrals of motion

$$p_0(q, p, t) = p \quad ; \quad q_0(q, p, t) = q - pt, \quad (2)$$

where the current position and momentum read :  $q = q_0 + p_0 t, p = p_0$ .

In case of position and momentum fluctuations, due for example to the temperature of the environment where the particle moves, the state of the particle is defined by the probability density  $f(q, p, t)$ , which provides for any time  $t$  the probability to find the particle in the phase space region  $dqdp$  around the point  $(q, p)$  :

$$dw(q, p, t) = f(q, p, t) dqdp. \quad (3)$$

Of course, such a probability density is nonnegative and normalized for any time  $t$  :

$$f(q, p, t) \geq 0 \quad ; \quad \int f(q, p, t) dqdp = 1. \quad (4)$$

The probability density is known to satisfy the Liouville kinetic equation, that for any Hamiltonian with potential  $V(q)$  reads as:

$$\left[ \frac{\partial}{\partial t} + p \frac{\partial}{\partial q} - \frac{\partial V(q)}{\partial q} \frac{\partial}{\partial p} \right] f(q, p, t) = 0. \quad (5)$$

The solutions of this equation can be expressed in terms of two integrals of motion,  $q_0(q, p, t)$  and  $p_0(q, p, t)$ , which have the physical meaning of providing the initial point of the particle trajectory in phase space, so that:

$$q_0(q, p, t = 0) = q \quad ; \quad p_0(q, p, t = 0) = p. \quad (6)$$

Thus we have for the solution  $f(q, p, t)$  of the Cauchy problem corresponding to those data the expression

$$f(q, p, t) = f_0(q_0(q, p, t), p_0(q, p, t)). \quad (7)$$

Here  $f_0(q, p)$  is an initial probability density in the particle phase space. All the statistics of position and momentum can be given by taking derivatives of the characteristic function, defined as

$$\chi(k_1, k_2, t) = \langle \exp[i(k_1 q + k_2 p)] \rangle_f = \int \exp[i(k_1 q + k_2 p)] f(q, p, t) dqdp. \quad (8)$$

For a free particle we have the Liouville equation

$$\left[ \frac{\partial}{\partial t} + p \frac{\partial}{\partial q} \right] f(q, p, t) = 0, \quad (9)$$

whose solution corresponding to the initial conditions of eq. (2) reads

$$f(q, p, t) = f_0(q - pt, p). \quad (10)$$

The propagator providing the solution in the form

$$f(q, p, t) = \int K(q, p, q', p', t) f_0(q', p') dq' dp' \quad (11)$$

is given by a product of Dirac's delta functions

$$K(q, p, q', p', t) = \delta(q' - q + pt) \delta(p' - p). \quad (12)$$

In the case when fluctuations are absent we may choose as initial density

$$f_0(q, p) = \delta(q - \bar{q}) \delta(p - \bar{p}), \quad (13)$$

with  $\bar{q}$  and  $\bar{p}$  initial position and momentum of the free particle trajectory. Then eqs. (12), (13) provide the probability density at any time  $t \geq 0$ , also as a product of Dirac's delta functions

$$f(q, p, t) = \delta(\bar{q} + pt - q) \delta(\bar{p} - p). \quad (14)$$

So, the density time evolution corresponds completely to a trajectory determined by the classical Newton's law of motion.

## 2.2 Quantum motion

The quantum motion of a free particle in a standard formulation of quantum mechanics may be described by the Schrödinger evolution equation for the wave function  $\psi(q, t)$  when the environment influence is absent.

In general, for the Hamiltonian operator (hereafter  $\hbar = 1$ ) :

$$\hat{H} = \frac{\hat{p}^2}{2} + V(\hat{q}), \quad (15)$$

one has the equation

$$i\frac{\partial}{\partial t}\psi(q, t) = \left[-\frac{1}{2}\frac{\partial^2}{\partial q^2} + V(q)\right]\psi(q, t). \quad (16)$$

In presence of an environment, the quantum state will be described by a density operator  $\hat{\rho}$ , obeying the Von Neumann evolution equation, which in the position representation reads as

$$i\frac{\partial}{\partial t}\rho(q, q', t) = \left[-\frac{1}{2}\left(\frac{\partial^2}{\partial q^2} - \frac{\partial^2}{\partial q'^2}\right) + (V(q) - V(q'))\right]\rho(q, q', t). \quad (17)$$

Here  $\rho(q, q', t) = \langle q|\hat{\rho}(t)|q'\rangle$  is the density matrix in the position representation.

For a free particle, the Schrödinger evolution equation becomes

$$i\frac{\partial}{\partial t}\psi(q, t) = -\frac{1}{2}\frac{\partial^2}{\partial q^2}\psi(q, t), \quad (18)$$

while the Von Neumann evolution equation reads

$$i\frac{\partial}{\partial t}\rho(q, q', t) = -\frac{1}{2}\left(\frac{\partial^2}{\partial q^2} - \frac{\partial^2}{\partial q'^2}\right)\rho(q, q', t). \quad (19)$$

The position statistics of the quantum particle is expressed as

$$\langle \hat{q}^n(t) \rangle_\rho = \text{Tr}[\hat{\rho}(t)\hat{q}^n] = \int q^n \rho(q, q, t) dq. \quad (20)$$

Analogously, for the momentum we have

$$\langle \hat{p}^n(t) \rangle_\rho = \text{Tr}[\hat{\rho}(t)\hat{p}^n]. \quad (21)$$

Both these moments can be obtained by using the quantum characteristic function, that is the mean value of the usual displacement operator

$$\chi(k_1, k_2, t) = \langle \exp[i(k_1\hat{q} + k_2\hat{p})] \rangle_\rho. \quad (22)$$

In spite of the similarity of the above definition with the definition of the classical case, eq. (8), the quantum function cannot be considered a true characteristic function, corresponding to a probability density on phase space. In fact, as the position and momentum operators do not commute,

the covariance term of position and momentum may be given by eq. (22) only by taking into account the Heisenberg commutations relations,  $[\hat{q}, \hat{p}] = i$ . Due to uncertainty relations, in the quantum domain there does not exist any joint probability density of position and momentum.

For a free particle the Von Neumann evolution equation has the solution

$$\hat{\rho}(t) = \exp \left[ -\frac{i}{2} \hat{p}^2 t \right] \hat{\rho}(0) \exp \left[ \frac{i}{2} \hat{p}^2 t \right], \quad (23)$$

which corresponds to the classical solution of eq. (10).

If there is no environment influence, the state can be described by a normalized vector  $|\psi\rangle$  in a Hilbert space. For such a pure state, the corresponding density operator is the rank-one projector  $|\psi\rangle\langle\psi| = \hat{\rho}_\psi$ , and the formulae for position and momentum statistics read

$$\langle \hat{q}^n(t) \rangle_\psi = \text{Tr} [\hat{\rho}_\psi(t) \hat{q}^n] = \int q^n |\psi(q, t)|^2 dq \quad (24)$$

and

$$\langle \hat{p}^n(t) \rangle_\psi = \text{Tr} [\hat{\rho}_\psi(t) \hat{p}^n] = \int \psi^*(q, t) \left( -i \frac{\partial}{\partial q} \right)^n \psi(q, t) dq. \quad (25)$$

These formulae, in contrast to the classical ones obtained from the classical characteristic function of eq. (8), show explicitly that a joint position and momentum probability density cannot exist for a quantum particle.

The quantum propagator for a free particle gives the wave function at time  $t > 0$  in the form

$$\psi(q, t) = \int G(q, q', t) \psi(q', 0) dq', \quad (26)$$

where

$$G(q, q', t) = \frac{1}{\sqrt{2\pi i t}} \exp \left[ i \frac{(q - q')^2}{2t} \right]. \quad (27)$$

These formulae induce the formulae for the free particle propagator for time evolution of the density matrix in the position representation, which read correspondingly:

$$\rho(q, q', t) = \int K(q, q', q_0, q'_0, t) \rho(q_0, q'_0, 0) dq_0 dq'_0, \quad (28)$$

with

$$K(q, q', q_0, q'_0, t) = \frac{1}{2\pi t} \exp \left[ i \frac{(q - q_0)^2}{2t} - i \frac{(q' - q'_0)^2}{2t} \right]. \quad (29)$$

The above formula corresponds to the classical evolution of a probability density on phase space given by eq. (12). But, in contrast to this real classical phase space density, the above complex quantum propagator connects density states on the configuration space.

Moreover, in the quantum domain, the classical phase space eq. (4) is substituted by the nonnegativity and normalization conditions for the density operator:

$$\hat{\rho}(t) \geq 0 \quad , \quad \text{Tr} [\hat{\rho}(t)] = 1. \quad (30)$$

Nevertheless, the recently developed tomographic scheme [3] provides a unified description of both classical and quantum mechanics in terms of fair probability distributions of a random variable. The tomographic description is presented in the next section.

### 3 The tomographic representation

#### 3.1 Classical domain tomography

The tomographic representation of the states of a classical particle is introduced via Radon integral transform of the probability density  $f(q, p, t)$  on the particle phase space, i.e.

$$W(X, \mu, \nu, t) = \int f(q, p, t) \delta(X - \mu q - \nu p) dq dp \quad (31)$$

The recent development which provides the modification of Radon transform by using a window function instead of Dirac delta function is presented in [11].

The tomogram  $W(X, \mu, \nu, t)$  depends on the random variable  $X$  and two real parameters  $\mu, \nu$ . The random variable is a linear combination of position and momentum:

$$X = \mu q + \nu p \quad (32)$$

with

$$\mu = s \cos \theta \quad , \quad \nu = s^{-1} \sin \theta. \quad (33)$$

It may be interpreted as the particle position in a new phase space reference frame, with  $\theta$ -rotated and  $s$ -scaled  $q$ -axis. The tomogram defined by formula (31) is nothing but a marginal probability density done along the straight line given by eq. (32). Thus,

$$W(X, \mu, \nu, t) \geq 0 \quad , \quad \int W(X, \mu, \nu, t) dX = 1. \quad (34)$$

The previous Radon transform may be inverted:

$$f(q, p, t) = \frac{1}{4\pi^2} \int W(X, \mu, \nu, t) \exp[i(X - \mu q - \nu p)] dX d\mu d\nu, \quad (35)$$

so that probability densities on phase space and tomographic probability densities can be put into one-to-one correspondence. In other words, the introduced classical tomographic representation is equivalent to the classical phase space representation.

The evolution equation for the classical tomogram is obtained by Radon transforming the Liouville equation (5) and reads:

$$\left[ \frac{\partial}{\partial t} - \mu \frac{\partial}{\partial \nu} - \frac{\partial V(q)}{\partial q} \right]_{q \rightarrow -\frac{\partial}{\partial \mu} \left( \frac{\partial}{\partial X} \right)^{-1}} \nu \frac{\partial}{\partial X} W(X, \mu, \nu, t) = 0. \quad (36)$$

Please note that, in the force term  $-\partial V(q)/\partial q$ , the argument  $q$  is replaced by the operator  $-\frac{\partial}{\partial \mu} \left( \frac{\partial}{\partial X} \right)^{-1}$ . Explicitly, the operator  $\left( \frac{\partial}{\partial X} \right)^{-1}$  is defined in terms of a Fourier transform as

$$\left( \frac{\partial}{\partial X} \right)^{-1} \int f(k) \exp(ikX) dk = \int \frac{f(k)}{ik} \exp(ikX) dk. \quad (37)$$

Due to the presence of such a term for a generic potential, the evolution tomographic equation is integro-differential.

In view of the previous definition of the random variable, eq. (32), the tomograms  $W(X, 1, 0, t)$  and  $W(X, 0, 1, t)$  are the marginal probability distributions of position and momentum, respectively. So, the statistics may be written in unified way as

$$\langle X^n \rangle_{\mu, \nu, t} = \int X^n W(X, \mu, \nu, t) dX \quad (38)$$



This statistics may be obtained also by using the tomographic characteristic function

$$\chi(k, \mu, \nu, t) = \langle \exp(ikX) \rangle_{\mu, \nu, t} = \int \exp(ikX) W(X, \mu, \nu, t) dk. \quad (39)$$

The tomogram of a free particle described by the phase space density of eq. (10) is:

$$W(X, \mu, \nu, t) = W_0(X, \mu, \nu, t) = \int f_0(q - pt, p) \delta(X - \mu q - \nu p) dq dp. \quad (40)$$

In the propagating form of this equation,

$$W(X, \mu, \nu, t) = \int K(X, \mu, \nu, X', \mu', \nu', t) W_0(X', \mu', \nu', 0) dX' d\mu' d\nu',$$

the tomographic classical free motion propagator reads

$$K(X, \mu, \nu, X', \mu', \nu', t) = \delta(X - X') \delta(\mu - \mu') \delta(\nu + \mu t - \nu'). \quad (41)$$

Note that this propagator coincides with the one calculated by inserting eq. (35) into eq. (40), on the tomographic functions which are homogeneous of degree  $-1$ .

### 3.2 Quantum tomography

We preliminarily observe that the classical tomogram defined above may be written as a mean value:

$$\begin{aligned} W(X, \mu, \nu, t) &= \int f(q, p, t) \exp[ik(X - \mu q - \nu p)] \frac{dk}{2\pi} dq dp \\ &= \left\langle \int \exp[ik(X - \mu q - \nu p)] \frac{dk}{2\pi} \right\rangle_f. \end{aligned} \quad (42)$$

Then, the quantum tomogram may be introduced as operator valued version of the previous equation, i.e.

$$\begin{aligned} W(X, \mu, \nu, t) &= \left\langle \int \exp[ik(X - \mu \hat{q} - \nu \hat{p})] \frac{dk}{2\pi} \right\rangle_\rho \\ &= \text{Tr} \left[ \hat{\rho}(t) \int \exp[ik(X - \mu \hat{q} - \nu \hat{p})] \frac{dk}{2\pi} \right]. \end{aligned} \quad (43)$$

In other words, a quantum tomogram may be thought of as the quantum Radon transform of a quantum density state.

We remark that the importance of the quantum tomographic picture relies on the possibility to measure in quantum optical experiments [6] the above tomographic probability density.

The quantum tomogram is again the marginal probability density of the random variable  $X$ , corresponding to the spectral variable of the quantum Hermitian operator

$$\hat{X}_{\mu,\nu} = \mu\hat{q} + \nu\hat{p}. \quad (44)$$

Again, the tomograms  $W(X, 1, 0, t)$  and  $W(X, 0, 1, t)$  are the marginal probability distributions of position and momentum of the quantum particle, respectively. By the way, we note that the Robertson-Schrödinger uncertainty relations [12, 13] can be expressed in tomographic form [14, 15].

Notably, also the quantum version of the Radon transform can be inverted [3] for reconstructing the density state

$$\hat{\rho}(t) = \frac{1}{2\pi} \int W(X, \mu, \nu, t) \exp[i(X - \mu\hat{q} - \nu\hat{p})] dX d\mu d\nu. \quad (45)$$

In general, the statistics of the random variable  $X$  is given by the same classical formula (38) by using the quantum tomographic density.

The evolution equation of a quantum tomogram is obtained by a quantum Radon transform of the Von Neumann equation for the density operator  $\hat{\rho}(t)$ . It reads

$$\left[ \frac{\partial}{\partial t} - \mu \frac{\partial}{\partial \nu} + i \left( V(q) \Big|_{q \rightarrow -\frac{\partial}{\partial \mu} \left( \frac{\partial}{\partial X} \right)^{-1} + \frac{i}{2} \nu \frac{\partial}{\partial X}} \right) - \text{c.c.} \right] W(X, \mu, \nu, t) = 0. \quad (46)$$

It is worthy to note that the first approximation in the parameter  $\nu$  decomposition of potential  $V$  yields the tomographic version of the classical Liouville equation (36).

For a free quantum particle  $V(q) = 0$  and the above equation reduces to the classical one, which reads

$$\left[ \frac{\partial}{\partial t} - \mu \frac{\partial}{\partial \nu} \right] W(X, \mu, \nu, t) = 0. \quad (47)$$

Thus, the solution to the quantum tomographic evolution equation is given by the same classical propagator (41).

## 4 Comparison of quantum and classical tomograms

As we have shown, the classical and quantum states of a free particle are described by a fair tomographic probability density of a random variable  $X$ . This probability density  $W(X, \mu, \nu, t)$  is a nonnegative normalized function in both classical and quantum domains. There is another common property. Due to the homogeneity property of the Dirac's delta function  $\delta(\lambda x) = |\lambda|^{-1} \delta(x)$ , the tomogram satisfies the same homogeneity condition

$$W(\lambda X, \lambda \mu, \lambda \nu, t) = |\lambda|^{-1} W(X, \mu, \nu, t). \quad (48)$$

This condition reflects the scaling property of the tomograms, which can be expressed in the form of a differential equation for the tomogram:

$$\left[ X \frac{\partial}{\partial X} + \mu \frac{\partial}{\partial \mu} + \nu \frac{\partial}{\partial \nu} + 1 \right] W(X, \mu, \nu, t) = 0. \quad (49)$$

Thus, all the quantum and classical tomograms, given by their own evolution equations, have to satisfy also the above constraint equation. One can easily check that both classical Liouville and quantum Von Neumann equations in tomographic form, eqs. (36), (46), are compatible with the above constraint equation, because they are scaling invariant, i.e., homogeneous of degree zero. For a free particle this is even more apparent because the quantum and classical evolution equations coincide.

Thus, we have to clarify differences between classical and quantum tomograms. In general, from a group theoretical point of view, these differences appear for instance in the inverse Radon transform formulae, containing different representations of the Weyl-Heisenberg group discussed in [16].

For a free particle, the tomographic propagator is the same for both classical and quantum evolutions, so it is indifferent to the kind of tomogram it propagates. So, the differences can be discussed only for the initial tomograms. A geometric point of view can help for choosing as a tool for such a discussion the Schrödinger-Robertson uncertainty relations

$$\sigma_{qq}\sigma_{pp} - \sigma_{qp}^2 \geq \frac{1}{4} \quad (\hbar = 1)$$

instead of the Heisenberg ones, because of their symplectic invariant character [17, 18]. This means that the left hand side in the above inequality is a

constant of motion. For instance, we can choose as initial state the Gaussian tomogram

$$W_0(X, \mu, \nu, t = 0) = \frac{1}{\sqrt{2\pi\sigma_{XX}(\mu, \nu)}} \exp\left[-\frac{X^2}{\sigma_{XX}(\mu, \nu)}\right], \quad (\langle X \rangle_{\mu, \nu} = 0), \quad (50)$$

where

$$\sigma_{XX}(\mu, \nu) = \mu^2\sigma_{qq} + \nu^2\sigma_{pp} + 2\mu\nu\sigma_{qp}. \quad (51)$$

The initial covariance  $\sigma_{qp}$  can be taken to be zero. The chosen initial free particle tomogram yields a classical evolution if it does not satisfy the Schrödinger-Robertson uncertainty relations. So, if  $\sigma_{qq}\sigma_{pp} < 1/4$ , the evolution will be classical and for any  $t > 0$ ,  $\sigma_{qq}\sigma_{pp} - \sigma_{qp}^2 < 1/4$ . If the Schrödinger-Robertson inequality is satisfied, the evolution can be either classical or quantum. In other words, quantum and classical tomographic domains have a nonempty intersection.

These considerations may be made more precise [19] from a group theoretical point of view. Here, we limit to state only that necessary and sufficient condition for a tomographic function to be quantum is that the associated group function on Weyl-Heisenberg group is of positive type according to Naimark definition [20]. This means that, if  $\mu, \nu, \tau$  parametrize the Weyl-Heisenberg group unitary irreducible representation

$$U(\mu, \nu, \tau) = e^{i\tau} \exp[i(\mu\hat{q} + \nu\hat{p})], \quad (52)$$

the function  $\varphi_t$  defined on the group as

$$\varphi_t(\mu, \nu, \tau) = \text{Tr}[\hat{\rho}(t)U(\mu, \nu, \tau)] = e^{i\tau} \int \exp(iX) W(X, \mu, \nu, t) dX \quad (53)$$

has to be of positive type, for any  $t \geq 0$ .

We recall that a function  $\phi$  on a group  $G$  is a positive type function if the following quadratic form is positive semi-definite:

$$\sum_{j,k=1}^n \lambda_j^* \lambda_k \phi(g_j^{-1}g_k) \geq 0, \quad (j, k = 1, \dots, n), \quad (54)$$

for any choice of  $n$  complex numbers  $\lambda_k$  and group elements  $g_k \in G$ , for any  $n \in \mathbb{N}$ .

That tomographic condition amounts, in a different picture of quantum mechanics, to the condition that the density state reconstructed out of a tomogram by eq. (45) is a nonnegative operator.

## 5 Conclusions

We resume the main points of our work. The example of a free particle was used to show that both classical and quantum states can be represented as fair tomographic probability densities. These densities give an alternative unified state description for both probability densities on phase space in classical domain and state vector or density operator in quantum domain. The evolutions of classical and quantum states are given by tomographic kinetic equations, equivalent to Liouville and to Von Neumann equations respectively. Statistical properties of position and momentum are provided by a unified formula using tomographic probability distribution for classical and quantum motion. The necessary and sufficient conditions for a tomogram to describe a quantum or classical state are recalled in group theoretical terms. As a consequence, we get more understanding of the geometrical meaning of the classical and quantum Radon transform.

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